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Series studies of the Potts model: III. The 3-state model on the simple cubic lattice

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Abstract. The finite-lattice method of series expansion has been used to extend low-temperature series for the partition function, order parameter and susceptibility of the 3-state Potts model on the simple cubic lattice to order z^{43} and the high-temperature expansion of the partition function to order v^{21}. We use the numerical data to show that the transition is first-order, and estimate the latent heat, the discontinuity in the magnetization, and a number of other critical parameters.

1. Introduction

This is the third in a series of papers in which we study the critical behaviour of the q-state Potts model in both two and three dimensions using series expansions derived from the finite-lattice method. The first paper (Guttmann and Enting 1993a), denoted I hereafter, gave the general expressions used to derive high- and low-temperature expansions for the q-state Potts model. In I, series expansions for the $q = 2$ (Ising) case on the simple cubic lattice were analysed. The second paper (Briggs *et al* 1994), denoted II hereafter, presented and analysed series for the bulk thermodynamic properties for Potts models on the square lattice for integer q ranging from 2 to 10. These were used to develop and test series analysis techniques that could distinguish between first-order and continuous transitions. The present paper considers the 3-state model on the simple cubic lattice.

After the initial paper by Potts (1952), the model attracted little attention for almost two decades. During the 1970's there was greatly renewed interest in the model, with new exact results, series studies and renormalization-group calculations and applications to phase transitions in surface films. A particular concern at that time was the failure of renormalization-group calculations to reproduce the exact results for the order of the transition in two dimensions. A review by Wu (1982) described much of the work on the Potts model.

Of even greater interest is the behaviour of the three-dimensional Potts model. As noted above, for the $q = 2$ (Ising) case, the low-temperature series and some high-temperature series have recently been extended in I. For $q = 3$, the three-dimensional Potts model is of particular interest as it is identical to the $Z(3)$ clock model. This in turn is the centre of $SU(3)$, so it is believed that the effective theory for Polyakov loops in finite-temperature $d = 4SU(3)$ lattice gauge theory should be in the same universality class as the three-dimensional 3-state Potts model. The Polyakov loops are the order parameter for the deconfinement transition that is thought to take place in QCD as the temperature is raised

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(hadronic quarks going over to a plasma of free quarks and gluons). The connection (Svetitsky and Yaffe 1982) is then that, if the *whole space* of $d = 3$, $q = 3$ Potts Hamiltonians has a first-order transition, then the above $SU(3)$ transition must also be first-order. By this we mean not just nearest-neighbour ferromagnetic couplings, but the whole space of arbitrary short-range couplings and multi-body interactions, with only the $Z(3)$ symmetry property. On the other hand if such a Potts model could be found which displayed a continuous transition, then the $SU(3)$ transition could be either first-order or continuous, but the particular Potts model would be a candidate for a universality class of second-order transitions in the $SU(3)$ gauge model. Our work in this paper is restricted to the isotropic, nearest-neighbour Potts model, so we cannot answer the general question of the nature of the $SU(3)$ phase transition—except to proffer the rather vague notion that a first-order transition appears more likely.

A number of earlier studies of the Potts model (Knak Jensen *et al* 1979, Nienhuis *et al* 1981, Blöte and Swendsen 1979, Kim and Joseph 1975, Kogut and Sinclair 1981, Enting 1974, Herrman 1979) claimed to see evidence of a first-order transition for the three-dimensional 3-state model, but these were based on Monte Carlo analyses of small lattices, or series work on short series. More recent, and more precise Monte Carlo studies by Fukugita *et al* (1990) supported the first-order results, as did the Monte Carlo study of Lee and Kosterlitz (1991), who gave evidence that the critical number of states $q_c = 2.45 \pm 0.1$, above which the transition is first-order. In order to look at more of the space of short-range couplings, Gavai and Karsch (1992) carried out a Monte Carlo study of the three-dimensional 3-state Potts model with mixed nearest-neighbour ferromagnetic and next-nearest-neighbour antiferromagnetic coupling. They found a lack of criticality in these extended Potts models. Furthermore, Alves *et al* 1991 also used high-precision Monte Carlo methods to conclude that the transition was first-order. Very recently, Bhanot *et al* (1993) used a method similar to ours to extend the Potts model series, and analysed the series to provide additional evidence for a first-order transition. After this work was completed, Vohwinkel (1993) showed how the shadow-graph method of Sykes (1965) could be used to extend the series even further than we have. However, Vohwinkel has only presented the magnetization series, and no analysis.

Salient QCD calculations include those of Bacilieri *et al* (1988) and of Brown and Christ (1988) on $d = 4SU(3)$, who gave conflicting results, with the former claiming to find evidence of a second-order phase transition, while the latter claimed to find evidence of a first-order transition. Cabasino *et al* (1989) argued that the evidence was equally good for a first-order or continuous transition, while Brown (1989) argued for a first-order transition. Another large-scale QCD Monte Carlo calculation was carried out by Fukugita *et al* (1989), who claimed to find strong evidence for a first-order transition.

In order to distinguish between first-order and continuous transitions we have analysed the series using differential approximants (Guttmann 1989) to effectively represent the series. The studies of the square lattice system in II gave a procedure for determining the order of the transition in this way. We have used these ideas to locate the critical temperature, and to identify the nature of the transition.

The layout of the remainder of the paper is as follows. In the next section we briefly describe the finite-lattice method and the nature of the results we have obtained therefrom. In section 3 we analyse the data. In section 4 we present a discussion of the results.

2. Series expansions from the finite-lattice method

The definitions and notation follow the usage of I (and II). The standard q-state Potts model

is defined on a lattice with each site having a 'spin' variable that takes on q possible values (denoted '0' to $q-1$). An energy ΔE is associated with each pair of interacting sites that are in different spin states, and an energy of 0 applies to pairs of interacting sites in the same state. We consider the simple cubic lattice, with each site interacting only with its six nearest neighbours. Each site not in state '0' has an additional field energy H.

The thermodynamic quantities can be derived from the partition function. We choose the normalization such that the state with all sites in state '0' has zero energy. In this normalization, the partition function is commonly denoted Λ.

We work in terms of the expansion variables $z = \exp(-\Delta E/kT)$, $\mu = \exp(-H/kT)$ and the high-temperature variable $v = (1-z)/(1+(q-1)z)$.

For the simple cubic lattice, the high-temperature expansion for the partition function takes the form

$$\Lambda = q^{-2}(1+(q-1)z)^3\Phi(v) \tag{1}$$

with

$$\Phi(v) = \sum_n a_n v^n = 1 + 3(q-1)v^4 + \ldots .$$

For the low-temperature expansion, we use a modified field variable $x = 1-\mu$ and truncate at order x^2 so that the partition function is expressed as

$$\Lambda = \Lambda_0 + x\Lambda_1 + x^2\Lambda_2 + \ldots . \tag{2}$$

The expansion of the zero-field partition function is written as

$$\Lambda_0 = \sum_n \lambda_n z^n. \tag{3}$$

The internal energy is given by

$$U = z\frac{\mathrm{d}\Lambda_0}{\mathrm{d}z}/\Lambda_0. \tag{4}$$

the order parameter by

$$M = 1 + \frac{q}{q-1}\frac{\Lambda_1}{\Lambda_0} = \sum_n m_n z^n \tag{5}$$

and the susceptibility by

$$\chi = 2\frac{\Lambda_2}{\Lambda_0} - \frac{\Lambda_1}{\Lambda_0} - \left(\frac{\Lambda_1}{\Lambda_0}\right)^2 = \sum c_n z^n. \tag{6}$$

Note that for $q \geqslant 3$ an additional 'transverse' susceptibility can be defined (Straley and Fisher 1973).

Previously, series expansions for the Potts model on the simple cubic lattice had been obtained by Straley (1974) (for low-temperature $\ln Z$, M and χ to z^{24} and high-temperature series quoted to v^{10}) and by Miyashita *et al* (1979) ($\ln Z$ with full field dependence to order z^{33} and full temperature dependence to μ^{11}). In 1993, Bhanot *et al* extended the free-energy and magnetization series to 39 terms, and the susceptibility series to 35 terms. Vohwinkel

(1993) has given the magnetization series to 56 terms, and has (unpublished) similar length series for other thermodynamic properties.

The description of the finite-lattice calculations in I was couched in terms of the general q-state Potts model and it was that formalism that was used in the present study. As noted in I (see also II, equation (17*a*) and (17*b*)), the amount of memory required increases with q. The present calculations used cuboids with cross sections of up to 3×4 sites, giving the low-temperature series correct to z^{41}. As described in I, comparing the 3×4 approximation for $q = 2$ to higher-order $q = 2$ calculations allows us to determine the correction required to the 3×4 approximation for all other q values. This has enabled us to extend the low-temperature series to z^{43}.

For the high-temperature series, we ran on a 4×4 lattice, which required storage of 900 MByte. This gave series correct to v^{21}. All programs were run on an IBM 3090/400J with $\frac{1}{2}$ GByte of memory and 2 GByte of backing storage. One run was also performed on a Cray EL. The runs for the high-temperature series took about 50 hours; the low-temperature 3×4 lattice runs substantially less. To extend the low-temperature series would currently take 4 GByte of memory. However, for low-temperature series, we believe that the shadow-lattice method is computationally superior, so there seems little incentive to develop this method further for low-temperature Potts series. We discuss this point further in Guttmann and Enting (1993b).

The coefficients for λ_n, m_n, c_n and a_n for $q = 3$ are listed in tables 1 and 2. These series disagree in several places with previously published series. The low-temperature series disagree at z^{24} with those published by Straley (1974). This error had previously been detected using series obtained by the method of partial generating functions (Enting, unpublished) and in fact the error was a typographical mistake; the analysis by Straley used the correct series (Straley, personal communication). Apparently the term quoted in the appendix to Straley (1974) as $486(y_1^6 + y_2^6)x^{24}$ should have been $496(y_1^6 + y_2^6)x^{24}$. The low-temperature series also disagree with those of Miyashita *et al* (1979) at order z^{30}. We have not been able to obtain the full field-dependent corrections to their series, but note that their coefficient for $z^{30}\mu^5$ should be $(q-1)^5$ times the corresponding Ising coefficient (see Sykes *et al* 1965), i.e. it should be $44998\frac{2}{5}$ and not 45008, as published. As our coefficients agree with those of Bhanot *et al* and Vohwinkel, we are confident that they are correct.

The most serious discrepancies are between our new series and the high-temperature series published by Straley (1974), disagreeing at order v^8 and v^{10}. In view of the gross disagreements at quite low order, we present, in an appendix to this paper, an independent re-calculation of these series for general q, using a conventional weak-graph expansion. This expansion confirms our finite-lattice calculations and also reproduces the series for the mean number of clusters in bond percolation. The expansion also agrees with Ising model series from I, but this is a weaker test of either our weak-graph expansion or the finite-lattice method expansion because many graph types have zero contribution for $q = 2$ and, in particular, only even powers of v occur. Nevertheless, this appendix provides useful series for general q, and is likely to be useful for other workers as a check on any long series that may subsequently be obtained.

3. Analysis of series

The series generated as described above were all analysed by the method of differential approximants (DA) (Guttmann 1989, page 83ff). This method generalizes Padé approximants

Table 1. Coefficients in low-temperature expansions for Λ_0, M and χ, defined by equations (3), (5) and (6).

n	λ_n	m_n	c_n
0	1	1	0
6	2	−3	2
10	6	−18	24
11	6	−18	24
12	−12	42	−56
14	30	−135	270
15	60	−270	540
16	−96	477	−930
17	−132	648	−1296
18	346	−1980	4768
19	498	−2988	7968
20	−636	4140	−10560
21	−2210	14052	36922
22	3000	−21690	64812
23	7344	−52920	163440
24	−7110	55020	−165464
25	−25836	201852	−659088
26	17802	−162774	600024
27	107450	−914538	3278256
28	−59358	555750	−1980408
29	−353376	3229524	−12285816
30	105944	−1188327	5005014
31	1342914	−13301370	55200864
32	−77154	1402686	−6062712
33	−4995004	52334268	−227203096
34	−226914	95751	1954650
35	17383710	−195398208	914339736
37	−64127562	761838084	−3742275288
38	−32638848	359664885	−1761828642
39	231546628	−2910516786	15132717432
40	160963416	−1946958399	10380877350
41	−805061298	10681132140	−58385376120
42	−795051840	10207745148	−56515869708
43	2914349712	−40522674258	232316142012

by fitting an ordinary differential equation of the form

$$\sum_{i=0}^{m} Q_i(x) D^i f(x) = P(x)$$

(with D^i denoting $(\frac{d}{dx})^i$) to the available series terms. Here $Q_k(x) = \sum_{i=0}^{m_k} q_{ki} x^i$ and $P(x) = \sum_{i=0}^{m_0} p_i x^i$ are polynomials. We chose $q_{m0} = 1$, so that the origin is not a regular singular point. This allows integration of the differential equation starting at $x = 0$. (This then corresponds to logarithmic derivative Padé approximants when $m = 1$.) For magnetization series, homogeneous DAs ($P \equiv 0$) are often most useful. Generally, the degrees of Q_k and P are chosen to use all (or most) of the available series terms. In principle, any order of differential equations can be used, but first-order ($m = 1$) was mostly used in the current work. Finding the coefficients of Q_k and P reduces to the solution of a system of linear equations, but this system is often ill-conditioned, so that care must be taken in its solution.

Table 2. Coefficients in the high-temperature expansion for Φ, defined by equation (1).

n	a_n
0	1
4	6
6	44
7	36
8	402
9	688
10	4836
11	11 364
12	69 466
13	196 374
14	1 097 436
15	3 583 084
16	18 627 090
17	67 523 316
18	335 693 618
19	1 305 112 008
20	6 332 595 828
21	25 841 846 466

These differential equations were then integrated numerically to obtain estimates of the desired physical quantities. In all cases a number (up to 10) of DAs were integrated and the results averaged to obtain the means and standard deviations shown in the tables and graphs below. All calculations were performed in quadruple precision (approximately 34 decimal places), so that all series terms could be represented without loss of precision.

We performed the numerical integration with an extrapolation method of the Bulirsch–Stoer type, as described by Hairer (1987, section II.9).

The analysis in II involved applying these methods to square-lattice series for comparison with the exact results of Baxter (1973, 1982) to assess the extent to which the methods could distinguish between first-order and continuous transitions. We showed that we could clearly distinguish the order of the transition in all known cases.

Simple sequence transformations were used to generate the most appropriate series, and hence DA, for numerical integration. In general, if a quantity is believed to behave like T^k at the origin of integration, it is often useful to transform the series $\sum_{i=0}^{N} a_i T^i$ say, to $\sum_{i=0}^{N} a_i T^i / T^k$ so that the transformed function approaches a constant at the origin. Thus, for example, we worked with the series for χ/z^6, rather than χ itself. Similarly, instead of the magnetization $M = 1 - 3z^6 - 18z^{10} - 18z^{11} \ldots$, we worked with $M - 1 + 3z^6$. We now discuss our numerical results in greater detail.

3.1. Internal energy

We integrated the internal energy series $U(z)$ from $T = 0$ and $T = \infty$ until they crossed at T_c. The results are shown in figure 1, with the intersection region shown as an inset. It is perfectly clear already, from this graph alone, that the transition is first-order. This follows from the fact that the gradient is clearly discontinuous. The intersection of the two curves gives the critical temperature. From a range of several approximants, we find the intersection at $kT/J = 1.8168 \pm 0.0012$, which compares with Monte Carlo estimates of 1.8166 ± 0.0002 (Yamagata 1993), $1.816\,454 \pm 0.000\,032$ (Alves *et al* 1991), 1.8164 ± 0.0001 (Fukugita *et al* 1990), 1.8161 ± 0.0001 (Gavai *et al* 1989) and $1.816\,24 \pm 0.000\,06$ (Wilson and Vause 1987).

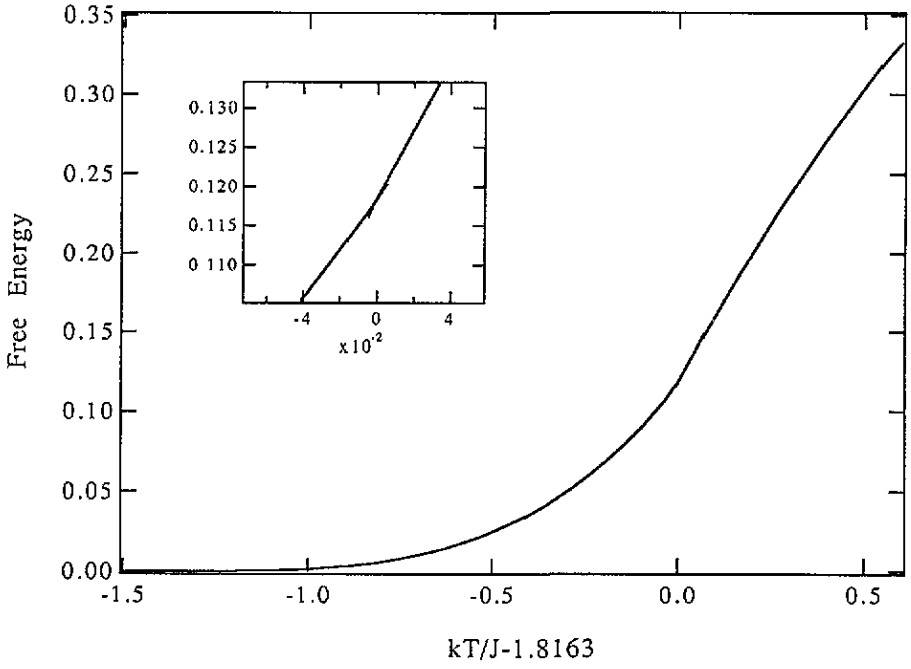

Figure 1. Free-energy versus $Tk/J - T_c k/J$, where the Monte Carlo estimate of $T_c k/J = 1.8163$ has been used. The inset figure shows a blow-up of the intersection of the high- and low-temperature curves.

It is clear that these estimates do not all agree within the stated precision. However, it is also clear that the Monte Carlo estimates are all slightly lower than our estimate. The average of the Monte Carlo estimates is taken to be 1.8163, and all subsequent analysis will be performed using both our central estimate of T_c and the Monte Carlo average. Note too that the MC estimate is made *under the assumption of a first-order transition*, while our series analysis makes no such assumption.

By differentiation, we can readily construct series for the high- and low-temperature internal energy. Integration of these gives a latent heat of 0.264 ± 0.011, where the error is one standard deviation in the average of the approximants. This error swamps the error induced by the uncertainty of the critical temperature. Our normalization of the Hamiltonian has, as a consequence, that the internal energy varies between 0 and 2. Other workers use a different normalization, in which the internal energy varies between 0 and 1. Therefore their latent heat estimates must be doubled to be compared with that given here. The specific heat at T_c^- is found to be 30 ± 4, while T_c^+ was considerably lower, at 11.1 ± 0.6. For a first-order transition, the specific heat is undefined at T_c, though the left- and right-hand limits are of course defined. By use of Monte Carlo methods, Gavai *et al* (1989) find a latent heat of 0.16 ± 0.008, Alves *et al* find 0.1606 ± 0.0006, while Gavai and Karsch (1992) find a latent heat of 0.160 ± 0.07. Our result is some 60% higher than these estimates. More precisely, we find $E(T_c^-) = 1.151 \pm 0.009$ and $E(T_c^+) = 1.414 \pm 0.004$. We have no explanation for the difference between our results and the Monte Carlo results. However, we note that our methods did give the correct latent heat in the two-dimensional case. That is, all the exact values lay within the range defined by the set of approximants with extreme outliers removed.

3.2. Magnetization

As T_c^- is approached, the gradient decreases rapidly. Consequently, the magnetization gap depends critically on the estimate of the critical temperature. Using the MC value of T_c, we find $\Delta M = 0.498$, while using our estimate of T_c we find $\Delta M = 0.505$. Another factor is the length of the series. Extending the series by 13 terms, as has been done by Vohwinkel, gives $\Delta M = 0.463$ at the Monte Carlo value of T_c—a drop of 7%. A similar decrease is found using the extended series at the other value of T_c used.

Monte Carlo estimates of this quantity have been obtained by Gavai *et al* (1989), who find $\Delta M = 0.395 \pm 0.005$, only a little lower than our estimate.

In table 3 we show the location and residues of Dlog Padé approximants to the magnetization series. They indicate a critical temperature of $e^{J/kT_c} \approx 0.5785$, some 0.3% above the value we believe to be correct. A 'pseudo magnetization exponent' around 0.2 is also suggested. As shown in II, this is entirely consistent with the existence of a first-order transition.

Table 3. $q = 3$, Dlog Padé approximants to the simple-cubic lattice spontaneous magnetization, giving the location and residue of the 'pseudo critical point'.

N		$[N-1/N]$		$[N/N]$		$[N+1/N]$
16	0.578 04	(0.1990)*	0.578 22	(0.2008)*	0.578 70	(0.2054)
17	0.579 08	(0.2090)	0.578 52	(0.2037)*	0.578 55	(0.2040)
18	0.578 56	(0.2041)	0.578 38	(0.2025)*	0.578 56	(0.2041)*
19	0.578 57	(0.2042)*	0.578 55	(0.2040)*	0.579 34	(0.2031)*
20	0.578 48	(0.2033)	0.578 47	(0.2031)		

3.3. Susceptibility

As for the specific heat, one expects the susceptibility at a first-order transition to be undefined, while having well defined left- and right-hand limits. As we only have low-temperature susceptibility series, we can only find one limit. We find a large but finite value of the susceptibility at T_c^-, notably 13 ± 3 on the low-temperature side.

4. Discussion of results

Hamer *et al* (1992) have studied the quantum Hamiltonian version of the 3-state Potts model in $(2+1)$ dimensions. They find $\Delta M = 0.42 \pm 0.02$, and a latent heat jump of 0.21 or 0.24, depending on the lattice. These numerical values are unexpectedly close to those of the three-dimensional 3-state Potts model which is the subject of this study. These quantities are non-universal, so no agreement is expected. Thus we see that the three distinct methods of study, series analysis and Monte Carlo of the Potts model, and series analysis of the quantum analogue of the Potts model, give consistent results. That is, they all find a fairly weak first-order transition, with a small latent heat, but quite a large jump in the magnetization.

We have shown how the finite-lattice method can provide competitive series in three-dimensions, though the computational complexity ensures that state-of-the-art direct methods, well programmed, will eventually be superior. Also, as the dimensionality increases, the method becomes steadily worse (Guttmann and Enting 1993b).

We have also shown that series methods can provide not only qualitative information about the nature of a phase transition, but also quantitative estimates of critical parameters for first-order, as well as second-order phase transitions.

Our methods have still provided by far the longest extant high-temperature series, and this is needed to determine the nature of the phase transition, as well as to identify its location.

Appendix. Comparison with weak-graph expansions

As noted in section 2, our high-temperature series shows significant differences from that given by Straley (1974). Since the program used to produce our series (and those of I) is our first use of a site representation for calculating high-temperature series, these differences caused us some concern. In order to check our program we recalculated the free energy using a conventional weak-graph expansion.

The graphs that are required are those with no vertices of order 1 and in which the removal of one line does not increase the number of components. Each graph requires a q-dependent weighting factor. This can be assigned by assigning each directed bond an integer flow in the range 1 to $q-1$. At each vertex the sum of inwards flows minus outwards flows must be 0 modulo q (see, for example, Straley 1974). The weight is the number of ways such flows can be assigned. An alternative algebraic formulation is given by Domb (1974). In fact, as discussed by Essam and Tsallis (1986), these are just flow polynomials, and they give all the polynomials for graphs with less than six independent cycles. Their data provides a check of our calculation of weights in table A1. For planar graphs, the weighting is equivalent to q^{-1} times the number of q-colourings of the graph and its exterior (see, for example, Wu 1982).

Table A1 gives details of the expansion. To save space, the lowest-order terms are not shown in the table. They are, at fourth order, three polygons, at sixth order, 22 polygons and at seventh order, 18 theta graphs. These then give the following low-order terms, with subsequent terms being given in the table:

$$\frac{\mathrm{d}}{\mathrm{d}q}\ln\lambda(q=1) = 3x^4 + 22x^6 - 18x^7 + \ldots$$
$$\ln\Lambda(2) = 3x^4 + 22x^6 + 0x^7 + \ldots$$
$$\ln\Lambda(3) = 6x^4 + 44x^6 + 36x^7 + \ldots$$
$$\ln\Lambda(4) = 9x^4 66x^6 + 108x^7 + \ldots$$
$$\ln\Lambda(5) = 12x^4 + 88x^6 + 216x^7 \ldots .$$

The results of the finite-lattice calculations agree with the weak-graph expansion for $\ln\Lambda$. Furthermore, the $q=2$ series (see I) reproduces the known Ising-model series. An interesting additional check is that our general-q expansion enables us to calculate $K(p)$, the mean number of clusters in the low-density regime of the bond percolation problem. Bond percolation can be regarded as the $q \to 1$ limit of the Potts model (see Fortuin and Kasteleyn (1972), or, more accesibly, Wu (1978)). Sykes (1986) gives series for site clusters which agree with our series, after taking into account the cluster of 1-site and zero bonds included in the Potts model limit, but excluded from conventional enumerations of bond percolation.

The agreement between the series of Sykes and the results derived from the weak-graph expansion of the Potts model give a useful test of our tabulation because the original derivation of the percolation series used a quite different type of expansion.

Table A1. Details of high-temperature expansion in terms of weak lattice constants. The body of the table gives the lattice constant for 8 to 14 bonds on the SC lattice. (The contribution from lower-order graphs is given in the appendix). Column 1 identifies graph type, in the notation of Sykes *et al* (1966). $(a.b)$ denotes a type a and type b graphs with a common vertex. $[a, b]$ is used to denote types a and b as separate components. The '8' denotes $(p.p)$, the 'figure 8' type. Column 2 is the weight, with a common factor of $q-1$ removed. Here S denotes $(q-1)$, Q denotes $(q-2)$, T denotes $(q-3)$, $W_F = Q(q^2-5q+7)$ and $W_{117} = Q(q^3-9q^2+29q-32)$, $W_{41} = Q(q^3-6q^2+149-13)$, $W_{69} = QT(q^2-4q+5)$, $W_{70} = Q^2(q^2-5q+8)$, $W_{78} = Q(q^3-7q^2+18q-17)$, $W_{79} = Q^2(q^2-5q+8)$, $W_{93} = (QT)^2$, $W_{99} = QT(q^2-5q+7)$, $W_M = Q(q^2-3q+3)$, $W_0 = (q^3-5q^2+10q-7)$, $W_Q = Q(q^2-2q+2)$.

Graph	W	8	9	10	11	12	13	14
p	1	207		2412		31754		452640
θ	Q	24	344	528	5934	12120	104250	239610
α	QT		8	24	228	996	5916	29448
β	Q^2			60	96	2556	6480	76752
γ	Q^2			84	288	2400	9552	52584
δ	Q^2+S			12		279		5388
F	W_F					15	168	1320
J	$Q(Q^2+1)$					48	192	1560
B	QT^3				12		384	756
117	W_{117}					1		24
G	Q^2T					96	336	4392
8=$(p.p)$	S	30		576		9306		152784
$(\theta.p)$	QS				456	648	14796	24900
$(8.p)$	S^2					458		14460
$[\theta.p]$	QS				−828	−1224	−28560	−49656
$[8.p]$	S^2					−1620		−54456
$[p, p]$	S	$\frac{-99}{2}$		−1020		−17510		−300108
$[p, p, p]$	S^2					1377		48642
$[\theta, \theta]$	Q^2S							−3384
$(\theta.\theta)$	Q^2S							1674
$[\alpha, p]$	QST							−408
$(\alpha.p)$	QST							216
$[\beta, p]$	Q^2S							−3450
$(\beta.p)$	Q^2S							1704
$[\gamma, p]$	Q^2S							−4956
$(\gamma.p)$	Q^2S							2604
$[\delta, p]$	Q^2S							−708
$(\delta.p)$	Q^2S							384
41	W_{41}							12
69	W_{69}							96
70	W_{70}							72
78	W_{78}							24
79	W_{79}							12
93	W_{93}							96
99	W_{99}							24
C	Q^2T						72	1344
D	Q^3						150	300
H	Q^3						600	2112
I	Q^3							264
K	Q^3						204	1248
L	Q^3						228	240
M	W_M						108	72
N	Q^3						144	1008
O	W_0							144
P	W_M						120	144
Q	W_Q						3	

Table A1. (continued)

Graph	W	8	9	10	11	12	13	14
$\frac{d}{dq} \ln \Lambda(q=1)$		183	−328	2034	−5142	26539	−81183	381222
$\ln \Lambda(2)$		$\frac{375}{2}$		1980		24044		319170
$\ln \Lambda(3)$		384	688	4572	11184	66158	190662	1050924
$\ln \Lambda(4)$		$\frac{1179}{2}$	2112	8856	34956	162624	693192	3153690
$\ln \Lambda(5)$		804	4320	15912	73224	358220	1573821	7815144

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